Lecture 11

Thursday, February 6, 2020 10:49 AM

- Finish pf of Thm 1 from Lecture 10 notes.

Cor 1. If $\Omega \subseteq \mathbb{C}^n$ is convex, then it is a d.o. holom.

Cor 2: If Ω_β, $\beta \in B$ ← any index set, are d.o. holom., then the interior of $\bigcap_\alpha \Omega_\alpha$ (if non-empty) is a d.o. holom.

Pfs of Cor 1-2: Ex.

Cor 3. Let $\Omega \subseteq \mathbb{C}^n$ be connected Reinhardt w/ $0 \in \Omega$. TFAE:

(i) Ω is d.o. conv. of a power series $\sum_\alpha a_\alpha z^\alpha$.

(ii) Ω is d.o. holom.

(iii) Ω is log. convex.

Pf. (i) ⇒ (iii) is the content of Thm already proved.

(ii) ⇒ (i) also follows Thm 1 (iii) and previously proved Thm.

(iii) ⇒ (ii). We shall show that $\hat{K}_\Omega \subset\subset \Omega$, $\forall K \subset\subset \Omega$. Thus, choose $K \subset\subset \Omega$. Since $\hat{K}_\Omega$ is bdd and closed in Ω, it suffices to show that the closure $\overline{\hat{K}_\Omega}$ in $\mathbb{C}^n$ is still contained in Ω. Pick $z \in \overline{\hat{K}_\Omega}$. We note, by assumption, if $\zeta \in \Omega$, then the closed polydisk $\overline{D^n_\zeta} = \{z \in \mathbb{C}^n : |z_j| \le |\zeta_j|,\ j = 1, \dots, n\} \subset\subset \Omega$. By compactness of K, we can find a finite set $F \subset\subset \Omega$ s.t.

$$K \subseteq \bigcup_{\zeta \in F} D^n_\zeta .$$

We may also assume that no $|\zeta_j| = 0$ for $\zeta \in F$. Go back to chosen $z \in \overline{\hat{K}_\Omega}$. WLOG $z \neq 0$ (since $0 \in \Omega$). By reordering the variables if necessary, we can find $l \in \{1, \dots, n\}$ s.t. $z_1 \cdots z_l \neq 0$, $z_j = 0$, $j > l$. By cont. (even if $z \in \partial\Omega$), we have $\forall \alpha \in \mathbb{Z}^l$

necessary, we can find $\ell \in \{1,\dots,n\}$ s.t. $z_1 \cdots z_\ell \neq 0$, $z_j = 0$, $j > \ell$.
By cont. (even if $z \in \partial\Omega$), we have $\forall \alpha \in \mathbb{Z}_+^\ell$

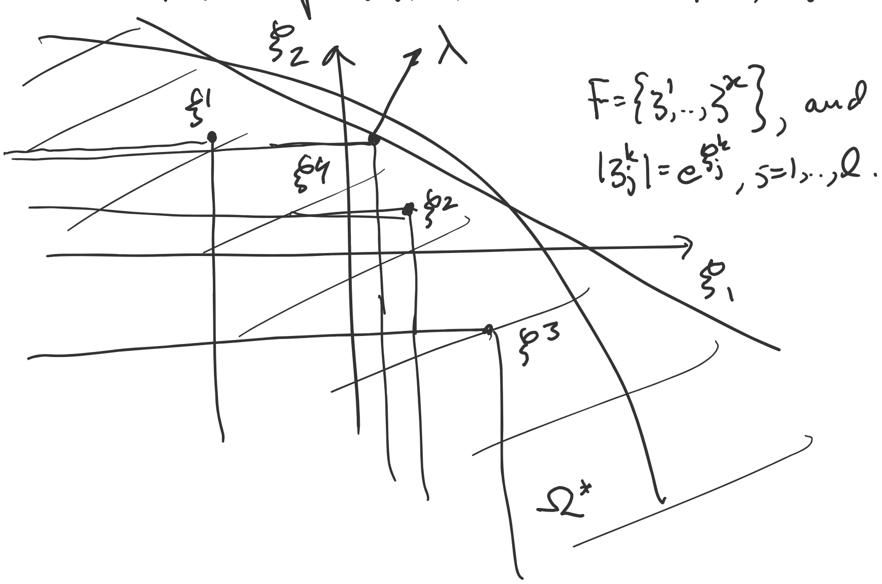
$$|z_1^{\alpha_1} \cdots z_\ell^{\alpha_\ell}| \le \max_{\zeta \in F} |\zeta_1^{\alpha_1} \cdots \zeta_\ell^{\alpha_\ell}|.$$

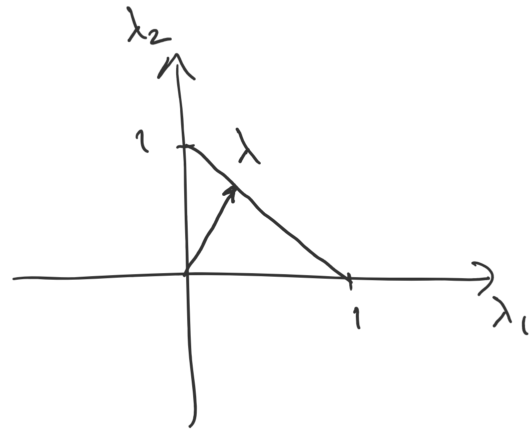
By taking log and letting $\lambda_j = \dfrac{\alpha_j}{|\alpha|}$, $j = 1, \dots, \ell$ $\Rightarrow$

$$\sum_{j=1}^{\ell} \lambda_j \log|z_j| \le \max_{\zeta \in F} \sum_{j=1}^{\ell} \lambda_j \log|\zeta_j| \qquad (1)$$

$\xi_j \to \xi' = (\xi_1, \dots, \xi_\ell) \in \mathbb{R}^\ell$

Since this holds for all rational $\lambda_1, \dots, \lambda_j \ge 0$, $\sum \lambda_j = 1$, it also holds for all real such $\lambda_1, \dots, \lambda_j$. [Pic. for $n=2$]:





Ex. Show that (1) $\Rightarrow \underbrace{(\log|z_1|, \dots, \log|z_\ell|)}_{\in \mathbb{R}^\ell}$ belongs to convex hull of all $\eta = (\eta_1, \dots, \eta_\ell) \in \mathbb{R}^\ell$ s.t. $\eta_j \le \xi_j^k$, $\xi_j^k = \log|\zeta_j^k|$, $j = 1, \dots, \ell$, $k = 1, \dots, x$, $F = \{\zeta^1, \dots, \zeta^x\}$.

Since the set E of such η sits in Ω^* (by assumpt.) and Ω^* is convex (by assumpt.), we conclude that $CH(E) \subseteq \Omega^* \Rightarrow z \in \Omega$.
Thus, the closure $\hat{K}_\Omega$ in $\mathbb{C}^n$ is contained in $\Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$. ☒

Thm 2. Let $\Omega \subseteq \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^m$ be d.o. holom., and $f: \Omega \to \mathbb{C}^m$ a holom. map. Then, $\Omega_0 := f^{-1}(\Omega') \subseteq \Omega$ is a d.o. holom.

Thm 2. Let $\Omega \subseteq \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^m$ be d.o. holom., and $f: \Omega \to \mathbb{C}^m$ a holom. map. Then, $\Omega_f := f^{-1}(\Omega') \subseteq \Omega$ is a d.o. holom.

Pf. Let $K \subset\subset \Omega_f$. Since $\Omega_f \subseteq \Omega$ and Ω d.o. holom. $\Rightarrow$ $\hat{K}_{\Omega_f} \subseteq \hat{K}_\Omega \subset\subset \Omega$. Suffices to show that closure $\overline{\hat{K}_{\Omega_f}}$ of $\hat{K}_{\Omega_f}$ in Ω is $\subseteq \Omega_f$. We let $K' = f(K) \subset\subset \Omega'$. Since Ω' d.o. holom., $\hat{K'}_{\Omega'} \subset\subset \Omega'$. Pick $z \in \overline{\hat{K}_{\Omega_f}}$, and $u \in \mathcal{O}(\Omega')$. Then, $u \circ f \in \mathcal{O}(\Omega_f)$, and so if $z^n \in \hat{K}_{\Omega_f}$, $z^n \to z \in \overline{\hat{K}_{\Omega_f}}$,

$$|u(f(z^n))| \le \sup_{z \in K} |u(f(z))| = \sup_{w \in K'} |u(w)|$$

$$\Rightarrow f(z^n) \in \hat{K'}_{\Omega'} \subset\subset \Omega' \Rightarrow f(z) = \lim_{n\to\infty} f(z^n) \in \hat{K'}_{\Omega'} \Rightarrow z \in \Omega_f.$$

QED